Lecture 12

<u>Thm 2.</u> Let $M \subseteq \mathbb{C}^n$ be CR mfd of CR dim $= m$. The following hold.

(i) $T_p^{1,0}M \cap T_p^{0,1}M = \{0\}$, $\forall p \in M$.

(ii) If Z, W are sections of $T^{1,0}M$, then $[Z, W]$ is a section of $T^{1,0}M$

(<u>Formal Integrability</u> "$[T^{1,0}M, T^{1,0}M] \subseteq T^{1,0}M$")

<u>Pf.</u> Both (i), (ii) follow from the corresp. property for $T^{1,0}\mathbb{C}^n$, since $T_p^{1,0}M = T_p^{1,0}\mathbb{C}^n \cap \mathbb{C}T_pM$. and the basic fact that if Z, W are vector fields tangent to M, then $[Z, W]$ is as well. Recall that $[Z, W]$ is the vector field given by

$$[Z, W]u = \underbrace{(ZW - WZ)}_{\text{nominally of order 2 but 2nd order derivatives cancel.}}u.$$

Commutators.

If $Z = \sum_j a_j \frac{\partial}{\partial z_j}$, $W = \sum_j b_j \frac{\partial}{\partial z_j}$ are sections of $T^{1,0}\mathbb{C}^n$ (thus, a_j, b_j are smooth functions on $\mathbb{C}^n$), then

$$[Z,W]f = (ZW - WZ)f =$$

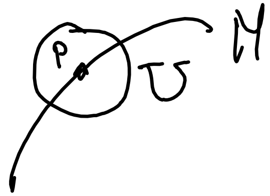
$$\sum_{i,j} \left(a_j \frac{\partial b_i}{\partial z_j} - b_j \frac{\partial a_i}{\partial z_j} \right) \frac{\partial}{\partial z_i} f.$$

Thus, it is clear that $[T^{1,0}\mathbb{C}^n, T^{1,0}\mathbb{C}^n] \subseteq T^{1,0}\mathbb{C}^n$. Property (ii) follows immediately. Prop. (i) for $T^{1,0}\mathbb{C}^n$ is obvious, so (i) for $T^{1,0}M$ follows immediately as well. $\boxtimes$

Complex mflds.

While the focus will be on real hypersurfaces, we briefly discuss the special case of CR mfld that are actually complex mflds.

A complex mfld of $\operatorname{codim}_{\mathbb{C}} = e$ (a particular case of a CR mfld of $\operatorname{codim}_{\mathbb{R}} = d = 2e$) is defined locally near each $p \in M$ by e holom functions $f_1, \dots, f_e$ w/ lin. independent $df_1, \dots, df_e$,



$$M \cap U = \{ z \in U : f_j(z) = 0, \ j = 1, \dots, e \}$$

It is easy to see that as a real submfld, $\dim_{\mathbb{R}} M = 2n - 2e = 2(n-e)$. We can take $\rho_1 = \operatorname{Re} f_1$, $\rho_2 = \operatorname{Im} f_1$, $\dots$ $\rho_{2e-1} = \operatorname{Re} f_e$, $\rho_{2e} = \operatorname{Im} f_e$

and CR dim $= n-e$, since

$\partial \operatorname{Re} f_j = i \partial \operatorname{Im} f_j = \frac{1}{2} df_j \Rightarrow$

$= \operatorname{rank}_{\mathbb{C}} \{\partial \rho_1, \partial \rho_2, \dots, \partial \rho_{2e}\} = \operatorname{rank}_{\mathbb{C}} \{df_1, \dots, df_d\}$

$= e$.

Thus, a complex submfld of $\dim_{\mathbb{C}} = n-e$ is a CR mfld of CR dim $= n-e$, and

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M$$

The converse is also true. This is a consequence of the Newlander–Nirenberg Thm; see below.

Abstract CR structures

A <u>CR structure</u> on a smooth mfld M is a complex subbundle $\mathcal{V}$ $(=T^{1,0}M)$ of $\mathbb{C}TM$ s.t.

(i) $\mathcal{V}_p \cap \overline{\mathcal{V}_p} = \{0\}$, $\forall p \in M$

(ii) "$[\mathcal{V},\mathcal{V}] \subseteq \mathcal{V}$", i.e. $\forall$ sections Z, W of $\mathcal{V}$, $[Z,W]$ is a section of $\mathcal{V}$.

CR dim $M = \dim_{\mathbb{C}} \mathcal{V}$. If $m = \dim_{\mathbb{R}} M$, $k = $ CR dim M, then $\dim_{\mathbb{C}} \mathbb{C}T_pM = m$ and $\dim_{\mathbb{C}} \mathcal{V} = 2k$. The difference $(\dim_{\mathbb{C}} \mathbb{C}T_p / \mathcal{V} \oplus \overline{\mathcal{V}})$ $d' = m - 2k$ is called <u>CR codimension</u> of M.

Note that $\mathcal{V} \oplus \overline{\mathcal{V}}$ is <u>real</u>, i.e. $X_p \in \mathcal{V} \oplus \overline{\mathcal{V}} \Rightarrow \overline{X_p} \in \mathcal{V} \oplus \overline{\mathcal{V}}$. Thus, $H = (\mathcal{V} \oplus \overline{\mathcal{V}}) \cap TM$ is real subbundle of TM (of $\dim_{\mathbb{R}} H = 2k$) and

$$\mathbb{C} \otimes H = \mathcal{V} \oplus \overline{\mathcal{V}}.$$